

THE RELATION BETWEEN REAL AW*-FACTORS AND ANTI-AUTOMORPHISMS OF INVOLUTIVE (I.E. WITH PERIOD 2) *- (COMPLEX) AW*-FACTORS

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ABSTRACT. The paper of the is to initiate the study of real AW*-algebras in the framework of the theory of real C*-algebras and W*-algebras. It happens that in some aspects real AW*-algebras behave unlike complex AW*-algebras and sometimes their properties are completely different also from corresponding properties of real W*-algebras. We prove that if the complexification $A + iA$ of a real C*-algebra A is a (complex) AW*-algebra then A itself is a real AW*-algebra. By modifying the Takenouchi's examples of complex non-W*, AW*-factors we show that there exist real non-W*, AW*-factors. The correspondence between real AW*-factors and involutive (i.e. with period 2) *-anti-automorphisms of (complex) AW*-factors is established. We give the decomposition of real AW*-algebras into types I, II and III similar to the case of complex AW*-algebras or W*-algebras. It is proved that if A is a real AW*-factor and its complexification $M = A + iA$ is also an AW*-algebra (and therefore an AW*-factor) then the types of A and M coincide.

KEYWORDS: AW*-algebra, C*-algebra, factor, involutive *-anti-automorphism, complex Hilbert space, commutant, complexification, linear *-automorphism, conjugate, bicommutant, quaternions algebra, projection, isomorphic.

1. INTRODUCTION

The theory of operator algebras was initiated in a series of papers by Murray and von Neumann in thirties. Later such algebras were called von Neumann algebras or W*-algebras. These algebras are self-adjoint unital subalgebras M of the algebra $B(H)$ of bounded linear operators on a complex Hilbert space H , which is closed in the weak operator topology. Equivalently M is a von Neumann algebra in $B(H)$ if it is equal to the commutant of its commutant (von Neumann's bicommutant theorem). A factor (or W*-factor) is a von Neumann algebra with trivial center and investigation of general W*-algebras can be reduced to the case of W*-factors, which are classified into types I, II and III.

Real operator algebra is a *-algebra consisting of bounded (real) linear operators on a real Hilbert space H . If it is closed in the weak operator topology we have real W*-algebra, and if it is uniformly closed (i.e. in the norm topology) then we come to the notion of the real C*-algebra.

In his monograph [7] Li Bing-Ren has set up the fundamentals of real operator algebras and gave a systematic discussion of the real counterpart for the theory of W*- and C*-algebras.

A slightly different (but almost the same up to *-isomorphism) definition of real W*-algebras was given by E. Størmer [13,14]: A real von Neumann algebra (or real W*-algebra) is a real *-algebra \mathcal{R} of bounded linear operators on a complex Hilbert space containing the identity operator $\mathbf{1}$, which is

closed in the weak operator topology and satisfies the condition $\mathfrak{R} \cap i\mathfrak{R} = \{0\}$. The smallest (complex) von Neumann algebra $U(\mathfrak{R})$ containing \mathfrak{R} coincides with its complexification $\mathfrak{R} \cap i\mathfrak{R}$, i.e. $U(\mathfrak{R}) = \mathfrak{R} \cap i\mathfrak{R}$. Moreover \mathfrak{R} generates a natural involutive (i.e. of order 2) *-antiautomorphism $\alpha_{\mathfrak{R}}$ of $U(\mathfrak{R})$, namely $\alpha_{\mathfrak{R}}(x + iy) = x^* + iy^*$, where $x + iy \in U(\mathfrak{R})$, $x, y \in \mathfrak{R}$. It is clear that $\mathfrak{R} = \{x \in U(\mathfrak{R}) : \alpha(x) = x^*\}$. Conversely, given a (complex) von Neumann algebra U and any involutive *-antiautomorphism α on U , the set $\{x \in U : \alpha(x) = x^*\}$ is a real von Neumann algebra in the above sense.

It is not difficult to see that two real von Neumann algebras generating the same (complex) von Neumann algebra are isomorphic if and only if the corresponding involutive *-antiautomorphisms are conjugate. Thus the study of the above real von Neumann algebra can be reduced to the study of pairs (U, α) , where U is a (complex) von Neumann algebra and α - its involutive *-antiautomorphism.

2. PRELIMINARIES

Let H be a complex Hilbert space, $B(H)$ denote the algebra of all bounded linear operators on H . The weak (operator) topology on $B(H)$ is the locally convex topology, generated by semi norms of the form: $\rho(a) = |(\xi, a\eta)|$, $\xi, \eta \in H$, $a \in B(H)$. W*-algebra is a weakly closed complex *-algebra of operators on a Hilbert space H containing the identity operator $\mathbf{1}$. Recall that W*-algebras are also called *von Neumann algebras*.

Let further M be a W*-algebra. The set M' of all elements from $B(H)$ commuting with each element from M is called the commutant of the algebra M . The center $Z(M)$ of a W*-algebra M is the set of elements of M , commuting with each element from M . It is easy to see that $Z(M) = M \cap M'$. Elements of $Z(M)$ are called *central* elements. A W*-algebra M is called *factor*, if $Z(M)$ consists of the complex multiples of $\mathbf{1}$, i.e. if $Z(M) = \{\lambda\mathbf{1}, \lambda \in \mathbb{C}\}$. We say that a W*-algebra M is *injective* if there exists a projection P in $B(H)$ onto M such that $\|P\| = 1$ and $P(\mathbf{1}) = \mathbf{1}$. This is equivalent to the condition that M is *hyperfinit*, i.e., that there exists an increasing sequence $\{M_n\}$ of matrix subalgebras of the algebra M containing $\mathbf{1}$ and such that the union $\cup_n M_n$ is weakly dense in M .

Let e, f, h be projections from M . We say that e is equivalent to f , and write $e \sim f$, if $e = \omega^* \omega$, $f = \omega \omega^*$ for some partial isometry ω from M . A projection e is called: *finite*, if $e \sim f \leq e$ implies $f = e$; *infinite* - otherwise; *purely infinite*, if e doesn't have any nonzero finite subprojection; *abelian*, if the algebra eMe is an abelian W*-algebra. A W*-algebra M is called *finite*, *infinite*, *purely infinite*, if $\mathbf{1}$ is a finite, infinite, purely infinite respectively; M is σ -finite, if any family of pairwise orthogonal projections from M is at most countable; *semifinite*, if each projection

in M contains a nonzero finite subprojection; *properly infinite*, if every nonzero projection from $Z(M)$ is infinite; *discrete*, or of type I , if it contains a faithful abelian projection (i.e. an abelian projection with the central support 1); *continuous*, if there is no abelian projection in M except zero; M is of type II , if M is semifinite and continuous; type I_{fin}^I (respectively I_{∞}^I), if M is of type I and finite (respectively properly infinite); type II_1^{II} (respectively type II_{∞}^{II}), if M is of type II and finite (respectively properly infinite); type III , if M is purely infinite. It is known that any W^* -algebra has a unique decomposition along its center into the direct sum of W^* -algebras of the $I_{fin}^I, I_{\infty}^I, II_1, II_{\infty}$ and III types.

A linear mapping $\alpha : M \rightarrow M$ is called a **-automorphism* (respectively a **-antiautomorphism*) if $\alpha(x^*) = \alpha(x)^*$ and $\alpha(xy) = \alpha(x)\alpha(y)$ (respectively $\alpha(xy) = \alpha(y)\alpha(x)$), for all $x, y \in M$. A mapping α is called *involutive* if $\alpha^2 = id$. A **-automorphism* α is called *inner* if there exists a unitary u in M , such that $Adu(x) = uxu^*$, for all $x \in M$. A **-automorphism* is called *centrally trivial* if $\alpha(x_n) - x_n \rightarrow 0$ **-strongly* as $n \rightarrow \infty$ for any central sequence $\{x_n\}_{n \in \mathbb{N}}$. We shall denote by $Aut(M)$ the group of all **-automorphisms*, by $Aut(M)$ the group of all **-antiautomorphisms*, by $Int(M)$ the group of all *inner *-automorphisms*, and by $Ct(R)$ the subgroup of its *centrally trivial *-automorphisms* of M . Two **-automorphisms* or **-antiautomorphisms* α and β are said to be *conjugate* (or *outer conjugate*), if $\alpha = \theta \cdot \beta \cdot \theta^{-1}$ (respectively $Adu \cdot \alpha = \theta \cdot \beta \cdot \theta^{-1}$) for some **-automorphism* θ (and an *inner *-automorphism* Adu). A linear functional ω on M is called *positive*, if $\omega(x^*x) \geq 0$ for all $x \in M$. A *positive linear functional* ω on M with $\|\omega\| = 1$ is called a *state*. Let M_+ be the *positive part* of M . A *weight* on M is a *homogeneous additive function* $\omega : M_+ \rightarrow [0, +\infty]$ (we suppose that $0 \cdot +\infty = 0$). A *weight* (or a *state*) ω is called: *faithful*, if for any $x \in M_+$, $\omega(x) = 0$ implies $x = 0$; *normal*, if for any net $\{x_\alpha\}$ in M , increasing to an element x , we have $\omega(x) = \sup_{\alpha} \omega(x_\alpha)$; *finite*, if $\omega(x) < \infty$ for all $x \in M_+$; *semifinite*, if for any $x \in M_+$ there exists a net of elements $\{y_\alpha\} \in M_+$, such that $\omega(y_\alpha) < \infty$, and $y_\alpha \rightarrow x$ in σ -weak topology; ω is a *trace*, if $\omega(uxu^*) = \omega(x)$ for all $x \in M_+$ and each unitary $u \in M$.

The type of a W^* -algebra is tightly connect with the existence of traces on it. Namely a W^* -algebra M is a *finite* if and only if it possesses a *separating family* of *finite normal traces*; it is *semifinite* if and only if it possesses a *faithful semifinite normal trace*; M is *purely infinite* if and only if there is no nonzero *semifinite normal trace* on M (see [15]).

Definition. [4]. By a *real C^* -algebra* we mean a *real Banach *-algebra* R such that the relation $\|a^*a\| = \|a\|^2$ holds and the element $1 + a^*a$ is *invertible* for any $a \in R$.

Definition*. [8,9]. A real C*-algebra R such that $R+iR$ is a complex W*-algebra is referred to as a real W*-algebra.

We proceed with another definition of a real W*-algebra, which can be found in papers of Størmer.

Definition.** [2,13]. A unital weakly closed real *-algebra R in $B(H)$ such that $R \cap iR = \{0\}$ is called a *real W*-algebra*.

A real W*-algebra R is called a (real) factor if its center $Z(R)$ consists of elements $\lambda \mathbf{1}$, $\lambda \in \mathbb{R}$. We say that a real W*-algebra R is of type $I_{fin}, I_\infty, II_1, II_\infty$ and III , $\lambda \in [0,1]$ if the enveloping W*-algebra $U(R) = R + iR$ (i.e., the least W*-algebra containing R) is of the corresponding type with respect to the usual classification of W*-algebras.

3. MAIR RESULTS

Let A be a real C*-algebra, with the complexification $M = A + iA$. Then M is a complex C*-algebra and, as we have seen in the previous section, if A is a real AW*-algebra M may not be a (complex) AW*-algebra. Now let us consider the converse problem if $M = A + iA$ is an AW*-algebra is A necessarily a real AW*-algebra? The following result gives a positive answer to this problem.

Proposition 1. *Let A be a real C*-algebra and let $M = A + iA$ be its complexification. Suppose that M is an AW*-algebra. Then A is a real AW*-algebra.*

Proof. As we have mentioned in the first section, A coincides with the fixed point set under the conjugate linear *-automorphism $\bar{\cdot} : x + iy \mapsto x - iy$ of M , where $x, y \in A$, i.e. $A = \{a \in M : \bar{a} = a\}$. If S is a nonempty subset in A then for its right-annihilator (with respect to M) we have

$$R_M(S) = \{a \in M \mid sa = 0 \text{ for all } s \in S\}$$

and

$$a \in R_M(S) \Leftrightarrow sa = 0, \forall s \in S \Leftrightarrow \overline{sa} = \bar{s} \bar{a} = s \bar{a} = 0, \forall s \in S$$

because $\overline{\bar{s}} = s \in A$. This means that $a \in R_M(S)$ if and only if $\bar{a} \in R_M(S)$.

Now suppose that M is an AW*-algebra, then $R_M(S) = gM$ for a suitable projection $g \in M$. Since $g \in R_M(S)$ from above it follows that $\bar{g} \in R_M(S)$. Therefore \bar{g} is a projection and $\bar{g} \in gM$, i.e. $\bar{g} = g\bar{g}$. Thus $(\bar{g})^* = (g\bar{g})^* = (\bar{g})^* g^* = \bar{g}g = \bar{g} = g\bar{g}$ i.e. $g = \bar{g} = \overline{g\bar{g}} = \overline{g\bar{g}} = \bar{g}$

This means that $g \in A$. But then

$$R_M(S) = R_M(S) \cap A = gM \cap A = gA,$$

i.e. A is a real AW*-algebra.

Proposition 2. *There exist real AW*-factors which are not real W*-factors.*

Theorem 1. *A real AW*-algebra A is a real W*-algebra if and only if*

- (i) A possesses a separating family of normal states;

(ii) its complexification $M = A + iA$ is an AW*-algebra.

Proof. Necessity is obvious, since if A is a real W*-algebra, then $M = A + iA$ is a (complex) W*-algebra (see [7, Chap.5]). Therefore, M is an AW*-algebra and it possesses a separating family of normal states, the restrictions of which on A give a separating family of normal states on A .

Sufficiency. Let $M = A + iA$ be an AW*-algebra and let A possess a separating family of normal states, which we denote by $\{f_\gamma\}$, i.e. for any $a \in A, a \geq 0, a \neq 0$ exists $f \in \{f_\gamma\}$ with $f(a) > 0$.

For $x \in a + ib \in M, a, b \in A$, we put $\alpha(x) = a^* + ib^*$. A straightforward calculation shows that α is an involutive (i.e. with period 2) *-anti-automorphism of M , and $A = \{a \in M : \alpha(a) = a^*\}$.

The extension of f_γ by linearity on M we denote by f_γ^0 , and we shall show, that the family $\{f_\gamma^0\}$ is a separating family of normal states on M .

For $x = a + ib \in M_s = \{x \in M : x^* = x\}$ we have $a^* = a, b^* = -b$, and since f_γ is hermitian we obtain $f_\gamma^0(x) = f_\gamma(a) + if_\gamma(b) = f_\gamma(a)$, since $f_\gamma(b) = 0$. Thus, for $x \in M_s$ we have

$$f_\gamma^0(x) = \frac{1}{2} f_\gamma(x + \alpha(x)),$$

and $x + \alpha(x) \in A$, since $x + \alpha(x) = 2a$.

If $x \geq 0$, then $\alpha(x) \geq 0$ (because α is a *-anti-automorphism), and hence $x + \alpha(x) \geq 0$, i.e.

$x + \alpha(x) \in A^+$. Therefore $f_\gamma^0(x) = \frac{1}{2} f_\gamma(x + \alpha(x)) \geq 0$, i.e. all functionals f_γ^0 are positive on M .

Moreover, we have $f_\gamma^0(1) = f_\gamma(1) = 1$, i.e. $\{f_\gamma^0\}$ is a family of states on M .

Now let us show, that each state f_γ^0 is normal. If $\{x_\nu\} \subset M$ is an arbitrary net with $x_\nu \searrow 0$, then since α is an order isomorphism of M , we have $\alpha(x_\nu) \searrow 0$. Therefore $x_\nu + \alpha(x_\nu) \searrow 0$ and $x_\nu + \alpha(x_\nu) \in A^+$. Since f_γ is a normal we obtain

$$f_\gamma^0(x_\nu) = \frac{1}{2} f_\gamma(x_\nu + \alpha(x_\nu)) \rightarrow 0$$

i.e. all functionals f_γ^0 are normal on M .

Finally, let $x \in M, x \geq 0$ and $f_\gamma^0(x) = 0$ for all γ . Then $x + \alpha(x) \in A^+$, and since $\{f_\gamma\}$ is a separating family of states, $x + \alpha(x) = 0$. Hence we have $x = -\alpha(x) \in M^+ \cap (-M^+) = \{0\}$, i.e.

$x = 0$. Thus, the AW*-algebra M possesses a separating family of normal states $\{f_\gamma^0\}$. By the theorem of Pedersen [10] M is a W*-algebra. Therefore, by [7] A is a real W*-algebra. Now, let M be a (complex) AW*-factor, α its involutive *-anti-automorphism. Then as it was mentioned above the set $A = \{\alpha \in M : \alpha(a) = a^*\}$ is a real C*-algebra such that $M = A + iA$ (actually $\bar{x} = \alpha(x^*)$ in terms of operation "-") and from Proposition 4.3.1 it follows that A is a real AW*-factor. It is known that two real W*-algebras generating the same (complex) W*-algebra, are isomorphic if and only if the corresponding involutive *-anti-automorphisms are conjugate [2,13,14]. A similar result is also valid for real AW*-algebras:

Proposition 3. *Let α and β be involutive *-anti-automorphisms of a (complex) AW*-factor M . Then the real AW*-factors*

$$A = \{x \in M : \alpha(x) = x^*\} \quad \text{and} \quad B = \{x \in M : \beta(x) = x^*\}$$

*are real *-isomorphic if and only if the involutive *-anti-automorphisms α and β are conjugate, i.e. $\beta = \theta\alpha\theta^{-1}$ for a suitable *-automorphism of the AW*-factor M .*

Proof. Let A and B be real *-isomorphic with a *-isomorphism $\theta_0 : A \mapsto B$. Then θ_0 can be naturally extended to a (complex) *-isomorphism θ of their complexifications $A + iA$ and $B + iB$ both coincide with M . Therefore θ is a *-automorphism of M and $\theta(A) = B$, i.e. $\alpha(x) = x^*$ if and only if $\beta(\theta(x)) = (\theta(x))^* = \theta x^*$. Thus for $x \in A$ we have

$$\beta(\theta(x)) = (\theta(x))^* = \theta(x^*) = \theta\alpha(x), \quad \text{i.e.} \quad \beta = \theta\alpha\theta^{-1}\alpha(x) \quad \text{for all } x \in A.$$

Since $\theta^{-1}\beta^{-1}\theta\alpha$ is a *-automorphism on M which is identical on A and any real *-automorphism of A can be uniquely extended to a complex *-automorphism of M , it follows that $\theta^{-1}\beta^{-1}\theta\alpha = id$ on whole M , i.e. $\theta\alpha = \beta\theta$ and $\beta = \theta\alpha\theta^{-1}$, i.e. α and β are conjugate.

Conversely, if α and β are conjugate, i.e. $\beta = \theta\alpha\theta^{-1}$ for a suitable complex *-automorphism θ of M , then $\theta\alpha = \beta\theta$ and $\alpha(x) = x^*$ if and only if $\beta(\theta(x)) = \theta x^* = (\theta x)^*$, i.e. $\theta(A) = B$. Therefore, θ restricted on A gives the needed *-isomorphism between real AW*-factors A and B .

Now we consider one of the main results of this section.

Theorem 2. *Let A be a real AW*-algebra and its complexification $M = A + iA$ is a (complex) AW*-algebra. Then A is of type I if and only if M of type I.*

Corollary. *Let A be a real AW*-algebra of type I, and its complexification $M = A + iA$ is a (complex) AW*-algebra. Then A is a real W*-algebra if and only if its center Z_A is a real W*-algebra.*

Proof. If A is a real W*-algebra then, obviously, its center Z_A is a real W*-algebra. Conversely, let A be an AW*-algebra of type I and its center is a W*-algebra. Then by Theorem 4.5.2, $M = A + iA$

is an AW*-algebra of type I, and its center $Z_M = Z_A + iZ_A$ is a W*-algebra. From Kaplansky's theorem [6, Theorem 2] it follows that M is a W*-algebra. Therefore, A is a real W*-algebra.

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